Recurrent perceptron (simplest recurrent NN)

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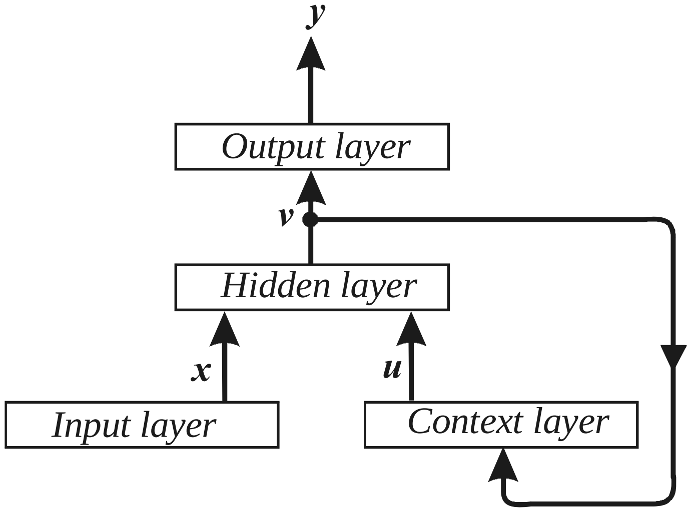
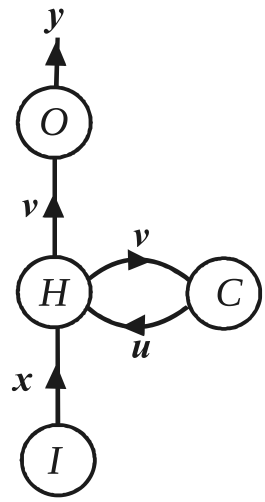
March 1999

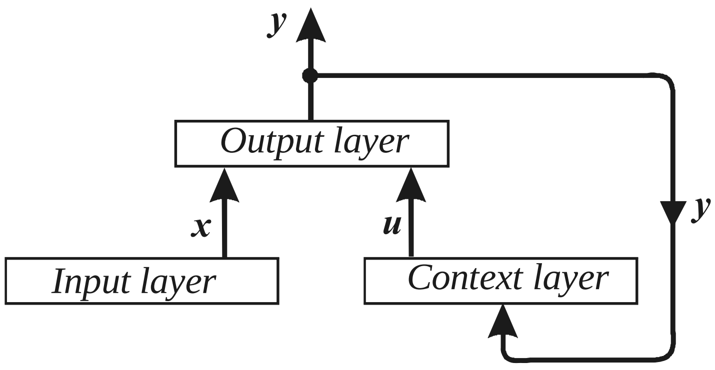
Modified Elman's RNN (recurrent perceptron)

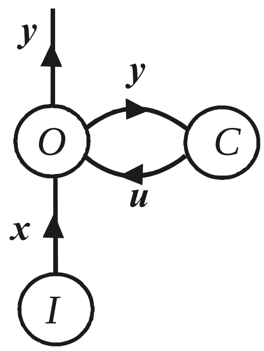
Standard architecture of Elman's recurrent neural network





This architecture is substantially simplified if the hidden neurons are removed, we arrive at the so-called recurrent perceptron (simples recurrent NN)





Activities of neurons are formally determined by a set of equations

$$\left.\begin{array}{l} \boldsymbol{x} = input \in \{a,b,c,d\}^{*} \\ \boldsymbol{u} = C(\boldsymbol{y}) \\ \boldsymbol{y} = O(\boldsymbol{x},\boldsymbol{u}) \in (0,1)^{n} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} \boldsymbol{x} = input \in \{a,b,c,d\}^{*} \\ \boldsymbol{y} = O(\boldsymbol{v}, C(\boldsymbol{y})) \in (0,1)^{n} \quad \text{(a nonlinearity)} \end{cases}$$

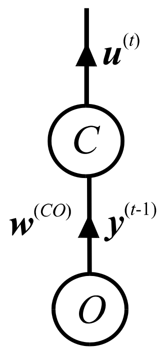
Their iterative solution ($x_{k+1}=f(x_k)$) specifies activities of neurons as follows

$$\left.\begin{array}{l} \boldsymbol{x}^{(t)} = \text{input} \\ \boldsymbol{u}^{(t)} = \begin{cases} 0 & (\text{for } t=1) \\ C\left(\boldsymbol{y}^{(t-1)}\right) & (\text{for } t \geq 2) \end{cases} \\ \boldsymbol{y}^{(t)} = O\left(\boldsymbol{x}^{(t)}, \boldsymbol{u}^{(t)}\right) \end{array}\right\} \quad \text{for } t = 1,2,\ldots,t_{\max}$$

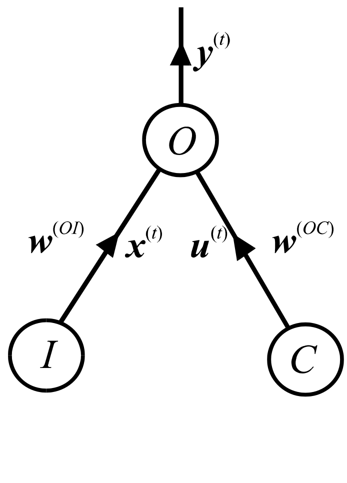
Activities are explicitly determined by

$$x_i^{(t)} = \text{input}$$

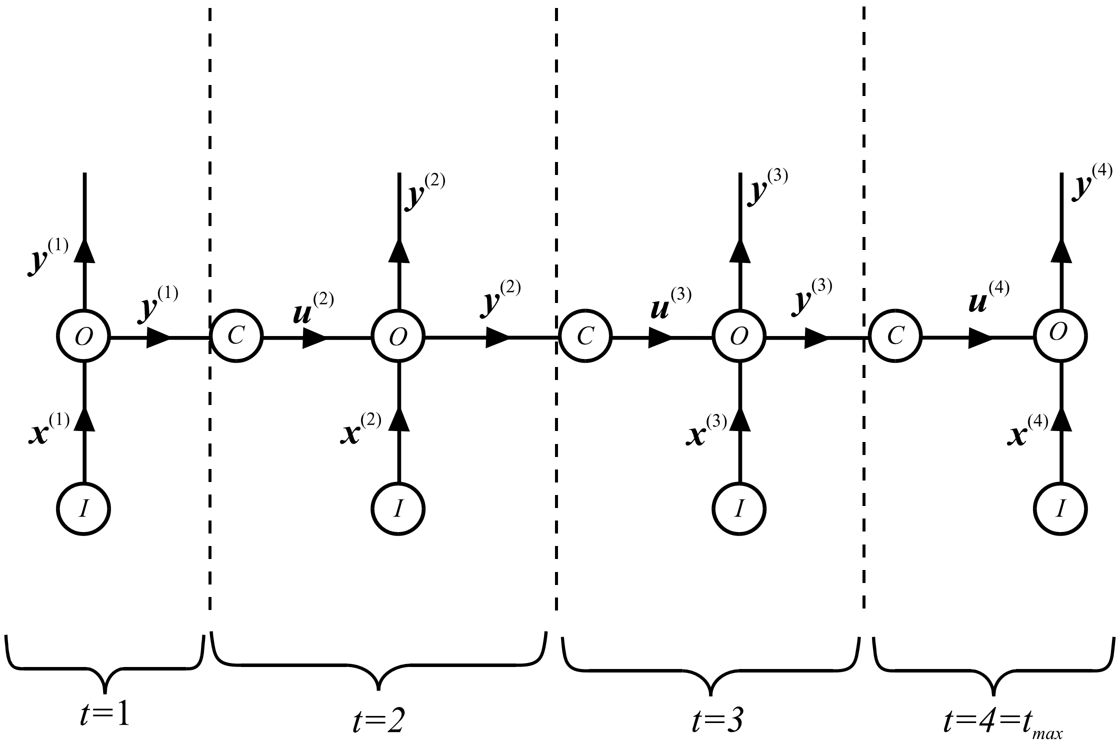
$$u_i^{(t)} = \begin{cases} 0 & (\text{for } t = 1) \\ t\left(\vartheta_i^{(C)} + \sum_{j=1}^{N_O} w_{ij}^{(CO)} y_j^{(t-1)}\right) & (\text{for } t > 1) \end{cases}$$



$$y_i^{(t)} = t\left(\vartheta_i^{(O)} + \sum_{j=1}^{N_I} w_{ij}^{(OI)} x_j^{(t)} + \sum_{j=1}^{N_C} w_{ij}^{(OC)} u_j^{(t)}\right)$$

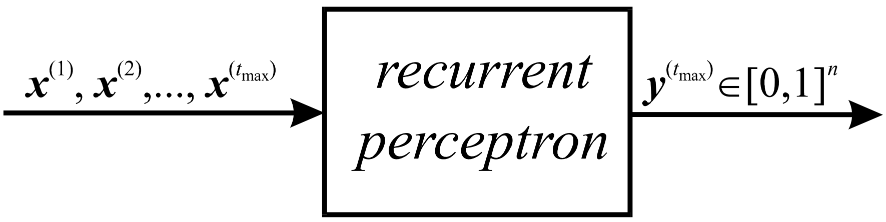


Unfolded recurrent perceptron



$$\mathbf{y}^{(t_{max})} = G\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)},\ldots,\mathbf{x}^{(t_{max})};\mathbf{w},\vartheta\right)$$

The recurrent perceptron can be understood as a parametric mapping of a sequence of inputs $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)},\ldots, \mathbf{x}^{(t_{max})}$ onto a real vector $\mathbf{y}^{(t_{max})} \in [0,1]^n$.



Adaptation (**learning**) of the recurrent neural network

$$A_{train} = \left\{ \left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(t_{max})} \right) / \left(\boldsymbol{y}_{req} \right) \right\}$$

$$E = \frac{1}{2} \left(\boldsymbol{y}^{(t_{max})} - \boldsymbol{y}_{req} \right)^2$$

An adaptation is equivalent to a minimization of the objective function E with respect to weight and threshold coefficients

$$\boxed{\left(w_{opt}, \vartheta_{opt} \right) = \arg \min_{(w,\vartheta)} E(w,\vartheta)}$$

This optimization problem is most frequently solved by the so-called **steepest descent** gradient method

$$w_{k+1} = w_k - \lambda \, grad \; E(\boldsymbol{x}_k)$$

In general, partial derivatives of the objective function E are determined as follows

$$\left(\frac{\partial E}{\partial \vartheta_i}\right) = t'(\xi_i)\left(g_i + \sum_k \frac{\partial E}{\partial \vartheta_k} w_{ki}\right)$$

$$t'(\xi_i) = t(\xi_i)\left[1 - t(\xi_i)\right]$$

$$g_i = \begin{cases} x_i - x_{i,req} & (i \in O) \\ 0 & (i \notin O) \end{cases}$$

$$\frac{\partial E}{\partial w_{ij}} = \frac{\partial E}{\partial \vartheta_i} x_j$$

These general formulae (they form a background of the so-called **back-propagation approach**) are immediately applicable to unfolded recurrent neural networks for calculation of partial derivatives of the objective function E.

(1) Initialization, t=$t_{\max}$

$$\left(\frac{\partial E}{\partial \vartheta_i^{(O)}}\right)^{(t_{max})} = y_i^{(t_{max})}\left(1-y_i^{(t_{max})}\right)\left(y_i^{(t_{max})}-y_{i,req}\right)$$

$$\left(\frac{\partial E}{\partial \vartheta_i^{(C)}}\right)^{(t_{max})} = u_i^{(t_{max})}\left(1-u_i^{(t_{max})}\right)\sum_{j=1}^{N_O}\left(\frac{\partial E}{\partial \vartheta_j^{(O)}}\right)^{(t_{max})} w_{ji}^{(OC)}$$

(2) Iteration, $1 \le t < t_{\max}$

$$\left(\frac{\partial E}{\partial \vartheta_i^{(O)}}\right)^{(t)} = y_i^{(t)}\left(1-y_i^{(t)}\right)\sum_{j=1}^{N_C}\left(\frac{\partial E}{\partial \vartheta_j^{(C)}}\right)^{(t+1)} w_{ji}^{(CO)}$$

$$\left(\frac{\partial E}{\partial \vartheta_i^{(C)}}\right)^{(t)} = u_i^{(t)}\left(1-u_i^{(t)}\right)\sum_{j=1}^{N_O}\left(\frac{\partial E}{\partial \vartheta_j^{(O)}}\right)^{(t)} w_{ji}^{(OC)}$$

Partial derivatives with respect to weight coefficients are determined by

(1) Initialization, $t=t_{\max}$

$$\left(\frac{\partial E}{\partial w_{ij}^{(OI)}}\right)^{(t_{max})}=\left(\frac{\partial E}{\partial \vartheta_i^{(O)}}\right)^{(t_{max})}x_j^{(t_{max})}$$

$$\left(\frac{\partial E}{\partial w_{ij}^{(OC)}}\right)^{(t_{max})}=\left(\frac{\partial E}{\partial \vartheta_i^{(O)}}\right)^{(t_{max})}u_j^{(t_{max})}$$

(2) Iteration, $1\leq t<t_{\max}$

$$\left(\frac{\partial E}{\partial w_{ij}^{(CO)}}\right)^{(t)}=\left(\frac{\partial E}{\partial \vartheta_i^{(C)}}\right)^{(t+1)}y_j^{(t)}$$

$$\left(\frac{\partial E}{\partial w_{ij}^{(OC)}}\right)^{(t)}=\left(\frac{\partial E}{\partial \vartheta_i^{(O)}}\right)^{(t)}u_j^{(t)}$$

$$\left(\frac{\partial E}{\partial w_{ij}^{(OI)}}\right)^{(t)}=\left(\frac{\partial E}{\partial \vartheta_i^{(O)}}\right)^{(t)}x_j^{(t)}$$

Total partial derivatives are determined as follows

$$\frac{\partial E}{\partial \vartheta_i^{(O)}} = \sum_{t=1}^{t_{max}} \left(\frac{\partial E}{\partial \vartheta_i^{(O)}} \right)^{(t)}$$

$$\frac{\partial E}{\partial \vartheta_i^{(C)}} = \sum_{t=2}^{t_{max}} \left(\frac{\partial E}{\partial \vartheta_i^{(C)}} \right)^{(t)}$$

$$\frac{\partial E}{\partial w_{ij}^{(OC)}} = \sum_{t=2}^{t_{max}} \left(\frac{\partial E}{\partial w_{ij}^{(OC)}} \right)^{(t)} = \sum_{t=2}^{t_{max}} \left(\frac{\partial E}{\partial \vartheta_i^{(O)}} \right)^{(t)} u_j^{(t)}$$

$$\frac{\partial E}{\partial w_{ij}^{(CO)}} = \sum_{t=1}^{t_{max}-1} \left(\frac{\partial E}{\partial w_{ij}^{(CO)}} \right)^{(t)} = \sum_{t=1}^{t_{max}-1} \left(\frac{\partial E}{\partial \vartheta_i^{(C)}} \right)^{(t+1)} y_j^{(t)}$$

$$\frac{\partial E}{\partial w_{ij}^{(OI)}} = \sum_{t=1}^{t_{max}} \left(\frac{\partial E}{\partial w_{ij}^{(OI)}} \right)^{(t)} = \sum_{t=1}^{t_{max}} \left(\frac{\partial E}{\partial \vartheta_i^{(O)}} \right)^{(t)} x_j^{(t)}$$